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Discontinuities of real–valued functions of one real variable

Abstract. The paper shows how to identify discontinuities of real–valued functions of one real variable and how to determine types of discontinuities. There are a few practical tasks with step–by–step solutions.

keywords: real–valued function of one real variable, types of discontinuities, continuity.

1. Definitions

The continuity of functions is a crucial concept in calculus but some types of discontinuities appear in many theorems so it is important to quickly identify the discontinuities. The theory in this paper is based on [2]. More practical tasks may be found in [1].

We consider a real–valued function of one real variable, i.e. $f:\ D_f \to \mathbb{R}$, $D_f \subset \mathbb{R}$.

Definition 1. *Function f has the discontinuity at x_0 if and only if exactly one of the following conditions holds:*

- *x_0 is the cluster point of D_f and $x_0 \notin D_f$,*
- *f is discontinuous at x_0.*

Point x_0 is then called the point of discontinuity of function f.

In the first case we can try to calculate the limit of f at x_0 (this limit exists or does not exist) but f is not defined at x_0. In the second case $x_0 \in D_f$ but the limit of f at x_0 does not exist or is different than $f(x_0)$. In three special cases, the discontinuities have their own names, i.e. removable discontinuity, finite jump, and infinite jump. Let us see their definitions.

Definition 2. *Function f has a removable discontinuity at x_0 if and only if $\lim\limits_{x\to x_0} f(x)$ exists as a finite value but it is different than $f(x_0)$ or function f is undefined at x_0.*

If a function has removable discontinuity at x_0, then we can easily define almost everywhere identical function which is continuous at x_0. A function with removable discontinuities is presented in Fig. 1.

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Data wpłynięcia: 10.07.2019 r.

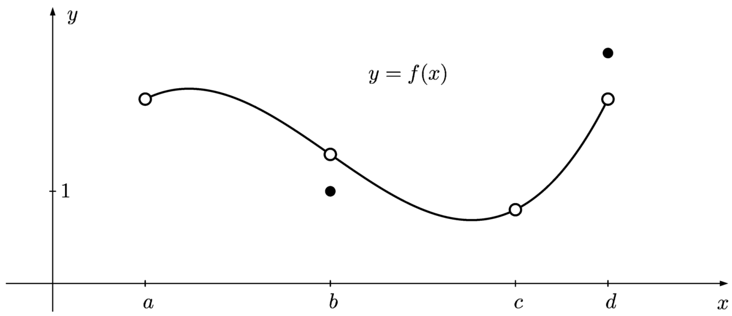

Fig. 1. The graph of function with removable discontinuities at a, b, c, and d

Definition 3. *Function f has a finite jump at x_0 if and only if both one–sided limits of f at x_0 are proper and different.*

Fig. 2 shows a function with three finite jumps.

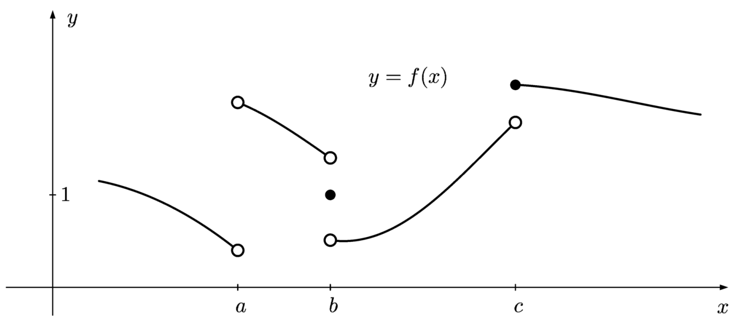

Fig. 2. The graph of function with finite jumps at a, b, and c

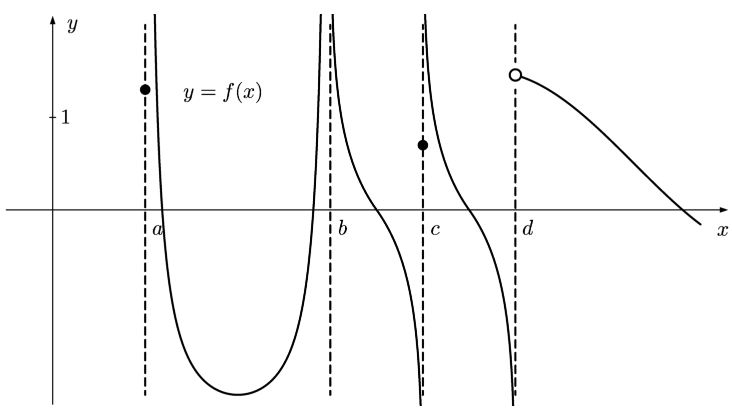

Fig. 3. The graph of function with infinite jumps at a, b, c, and d

Definition 4. *Function f has an infinite jump at x_0 if and only if at least one of one–sided limits of f at x_0 is improper.*

If function f has infinite jump at x_0, then line $x = x_0$ is the horizontal asymptote of curve $y = f(x)$. Infinite jumps are presented in Fig. 3.

Removable discontinuities and finite jumps are called *discontinuities of the I type*, other — *discontinuities of the II type*. Note that the class of discontinuities of the II type is very wide, it contains not only infinite jumps. For example, function $f(x) = \sin\frac{1}{x}$ has the II type discontinuity at 0 because both one-sided limits at 0 do not exist — in each deleted neighbourhood of 0 this function takes on infinitely many times all values form $[-1,1]$. Another popular example is the Dirichlet function:

$$D(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

If x_0 is any real number, then both $\lim\limits_{x\to x_0^-} D(x)$ and $\lim\limits_{x\to x_0^+} D(x)$ do not exist, so the Dirichlet function has the II type discontinuity at each point.

2. Examples

Example 1. Determine types of discontinuities of function

$$f(x) = \frac{\sin(x-3)}{x^2-4x+3}.$$

We have $f(x) = \dfrac{\sin(x-3)}{(x-3)(x-1)}$ so $D_f = \mathbb{R}\setminus\{1,3\}$. Function f is continuous on its domain (in the top we have the composition of two continuous functions, in the bottom — the polynomial).

Points 1 and 3 do not belong to domain but they are cluster points of domain so f has discontinuities at 1 and at 3. In order to determine their types, we have to calculate limits of f at these points.

We have

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{\sin(x-3)}{(x-3)(x-1)} \overset{[\frac{0}{0}]}{=} \lim_{x\to 3} \frac{\sin(x-3)}{x-3} \cdot \frac{1}{x-1} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

because if $x \to 3$ then $x-1 \to 2$ and $x-3 \to 0$ (we apply formula $\lim\limits_{t\to 0}\frac{\sin t}{t} = 1$) . Hence f has removable discontinuity (I type) at 3.

For the limit at 1 we have to calculate one–sided limits because there is not an indeterminate form (top tends to some number different than zero, bottom tends to zero). We obtain

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sin(x-3)}{(x-3)(x-1)} = \left[\frac{\sin(-2)}{-2\cdot 0^+}\right] = +\infty$$

because $\sin(-1) = -\sin 1 < 0$ (look at the graph of sine function) and if $x \to 1^+$ (which means that x is close to 1 and greater than 1) then $x-1$ tends to 0 (but is positive).

Analogously ($x \to 1^-$ means that x is close to 1 and less than 1):

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\sin(x-3)}{(x-3)(x-1)} = \left[\frac{\sin(-2)}{-2\cdot 0^-}\right] = -\infty.$$

Therefore f has infinite jump (the discontinuity of II type) at 1.

Remember!

A continuous function may posses discontinuities.

The graph of this function is presented in Fig. 4.

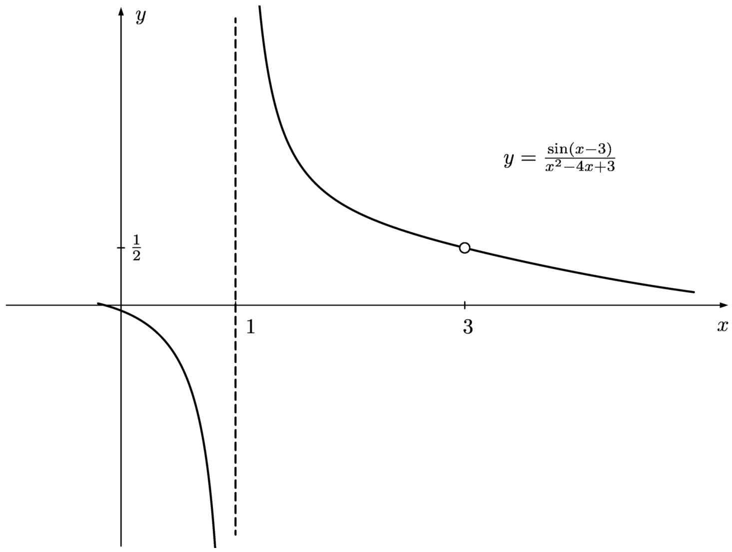

Fig. 4. The graph of function from Example 1

Example 2. Determine types of discontinuities of function

$$f(x) = \operatorname{arccot} \frac{x-1}{3-x}.$$

We see that $D_f = \mathbb{R} \setminus \{3\}$. Function f is continuous (as the composition of two continuous functions: arccot and the rational function). Hence f has the discontinuity at 3.

If $x \to 3^+$ (x is close to 3 but greater than 3), then:

- $x - 1 \to 2$
- $3 - x \to 0^-$
- $\frac{x-1}{3-x} \to -\infty$ because we have $[\frac{2}{0^-}]$
- $\operatorname{arccot} \frac{x-1}{3-x} \to \pi$

so $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \operatorname{arccot} \dfrac{x-1}{3-x} = \pi.$

If $x \to 3^-$ (x is close to 3 but less than 3), then:

- $x - 1 \to 2$
- $3 - x \to 0^+$
- $\frac{x-1}{3-x} \to +\infty$ because we have $[\frac{2}{0^+}]$
- $\operatorname{arccot} \frac{x-1}{3-x} \to 0$

so $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} \operatorname{arccot} \dfrac{x-1}{3-x} = 0.$

Therefore f has I type discontinuity (finite jump) at 3. The graph of this function is presented in Fig. 5.

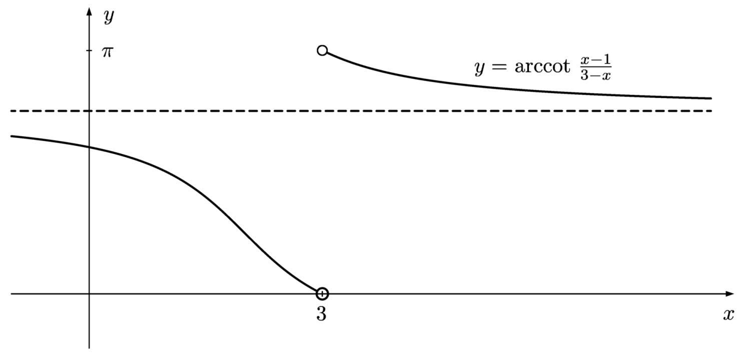

Fig. 5. The graph of function from Example 2

Example 3. Determine types of discontinuities of function

$$f(x) = \begin{cases} \sqrt{e} - (1-x)^{\frac{1}{x^2-2x}} & \text{if} \quad x < 0 \\ \arcsin x & \text{if} \quad x \in [0,1] \\ \ln(x-1) & \text{if} \quad x > 1. \end{cases}$$

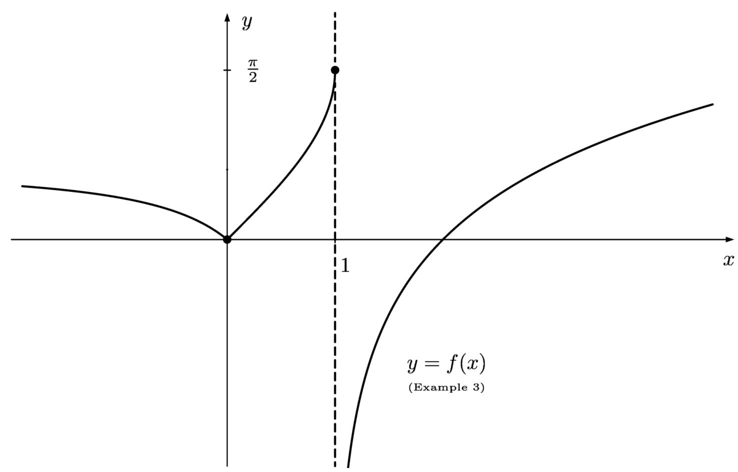

Fig. 6. The graph of function from Example 3

We see that $D_f = \mathbb{R}$. Function f is continuous on intervals: $(-\infty, 0)$, $(0, 1)$, $(1, +\infty)$. It may be discontinuous at 0 or 1, we have to check it.

Since

$$f(0) = \arcsin 0 = 0,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[\sqrt{e} - (1-x)^{\frac{1}{x^2-2x}}\right] = \lim_{x\to 0^-} \left[\sqrt{e} - \left[(1+(-x))^{-\frac{1}{x}}\right]^{\frac{-1}{x-2}}\right] = \sqrt{e} - e^{\frac{1}{2}} = 0,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \arcsin x = \arcsin 0 = 0,$$

function f is continuous at 0.

Since

$$f(1) = \arcsin 1 = \frac{\pi}{2},$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \arcsin x = \frac{\pi}{2},$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln(x-1) = \left[\begin{array}{l} u = x-1 \\ x\to 1^+ \Rightarrow u \to 0^+ \end{array}\right] = \lim_{u\to 0^+} \ln(u) = -\infty,$$

f has infinite jump at 1 (the discontinuity of II type).

The graph of this function is presented in Fig. 6.

Example 4. Determine types of discontinuities of function

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \in [-8, 0) \\ \sqrt{3} & \text{if} \quad x = 0 \\ -2 & \text{if} \quad x = \sqrt{2}. \end{cases}$$

We see that $D_f = [-8, 0] \cup \{\sqrt{2}\}$. Function f is continuous on interval $[-8, 0)$ and it is continuous at $\sqrt{2}$ (it is the isolated point of D_f).

Remember!

Each function is continuous at isolated points of its domain — it follows directly from the definition of continuity.

Function f may be discontinuous at 0, we have to check it. We obtain

$$f(0) = \sqrt{3},$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$

so f has removable discontinuity at 0 (we can redefine the function by the change of its value at 0 to obtain new function which is continuous at 0).

The graph of this function is presented in Fig. 7.

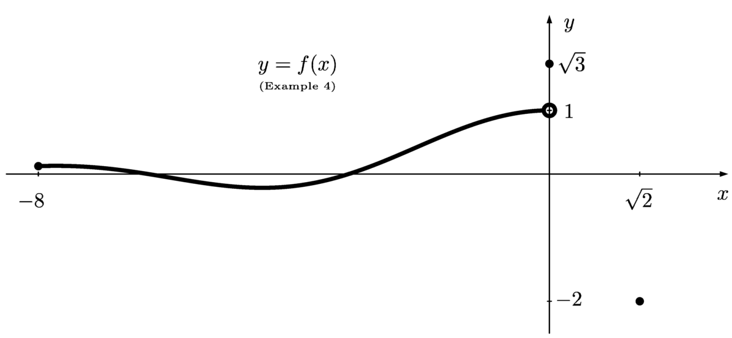

Fig. 7. The graph of function from Example 4

Example 5. Determine types of discontinuities of function

$$f(x) = \begin{cases} (5-x^2)^{\frac{x}{2-x}} & \text{if} \quad x \in [0, 2) \\ \frac{\sqrt{x^2+1}-1}{e^{x^3}-1} & \text{if} \quad x \in (-\infty, 0) \cup [2, +\infty). \end{cases}$$

We see that $D_f = \mathbb{R}$ and f is continuous on intervals $(-\infty, 0)$, $(0, 2)$, $(2, +\infty)$. The function may be discontinuous at points 0 or 2.

We have

$$f(0) = 5^0 = 1,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5 - x^2)^{\frac{x}{2-x}} = 5^0 = 1,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sqrt{x^2+1}-1}{e^{x^3}-1} = \lim_{x\to 0^-} \frac{\sqrt{x^2+1}-1}{e^{x^3}-1} \cdot \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} = \lim_{x\to 0^-} \frac{x^2}{(e^{x^3}-1)(\sqrt{x^2+1}+1)} =$$

$$= \lim_{x\to 0^-} \frac{x^3}{e^{x^3}-1} \cdot \frac{1}{x} \cdot \frac{1}{\sqrt{x^2+1}+1} = -\infty.$$

The last result was obtained because $\lim\limits_{x\to 0^-} \dfrac{x^3}{e^{x^3}-1} = \dfrac{1}{\ln e} = 1$ (we apply the formula $\lim\limits_{t\to 0} \dfrac{a^t - 1}{t} = \ln a$ for $a > 0$), $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$, $\lim\limits_{x\to 0^-} \dfrac{1}{\sqrt{x^2+1}+1} = \dfrac{1}{2}$.

Thus, f is discontinuous at 0 and there is II type discontinuity (infinite jump).

To verify the continuity of f at 2 we calculate:

$$f(2) = \frac{\sqrt{5}-1}{e^8-1},$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{\sqrt{x^2+1}-1}{e^{x^3}-1} = \frac{\sqrt{5}-1}{e^8-1},$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (5 - x^2)^{\frac{x}{2-x}} = \lim_{x\to 2^-} \left(1 + (4 - x^2)\right)^{\frac{x}{2-x}} = \lim_{x\to 2^-} \left[\left(1 + (4 - x^2)\right)^{\frac{1}{4-x^2}}\right]^{\frac{x(4-x^2)}{2-x}} =$$

$$= \lim_{x\to 2^-} \left[\left(1 + (4 - x^2)\right)^{\frac{1}{4-x^2}}\right]^{x(x+2)} = e^8.$$

Both one–sided limits of f at 2 are finite and different so f is discontinuous at 2 and there is I type discontinuity (finite jump).

Example 6. Determine types of discontinuities of function

$$f(x) = \begin{cases} \arccos\frac{1}{x} & \text{if} \quad x \neq 0 \\ 2\operatorname{arccot} x & \text{if} \quad x = 0. \end{cases}$$

First, we have to find the domain of f. Of course, we can calculate value of f at 0:

$$f(0) = 2\operatorname{arccot} 0 = 2 \cdot \frac{\pi}{2} = \pi.$$

Function arccos is defined for arguments from $[-1, 1]$ so we have (assuming $x \neq 0$):

$$-1 \le \frac{1}{x} \le 1$$

$$-1 \le \frac{1}{x} \quad \wedge \quad \frac{1}{x} \le 1$$

$$0 \le \frac{1}{x} + 1 \quad \wedge \quad \frac{1}{x} - 1 \le 0$$
$$0 \le \frac{1+x}{x} \quad \wedge \quad \frac{1-x}{x} \le 0$$
$$x \in (-\infty, -1] \cup (0, +\infty) \quad \wedge \quad x \in (-\infty, 0) \cup [1, +\infty)$$
$$x \in (-\infty, -1] \cup [1, +\infty).$$

By the way, it is worth to say that the double inequality $-1 \le \frac{1}{x} \le 1$ may be easily solved using graphs. We draw hyperbola $y = \frac{1}{x}$ and lines $y = -1$, $y = 1$. Then we look for points of hyperbola which are between these lines and read abscissae of these points.

Finally, the domain is $D_f = (-\infty, -1] \cup \{0\} \cup [1, +\infty)$. The function is continuous on intervals $(-\infty, -1]$, $[1, +\infty)$ as the composition of two continuous functions. It is also continuous at 0, which is the isolated point of D_f. Therefore, f is continuous and its domain is the closed set so function f does not posses discontinuities.

The graph of this function is presented in Fig. 8.

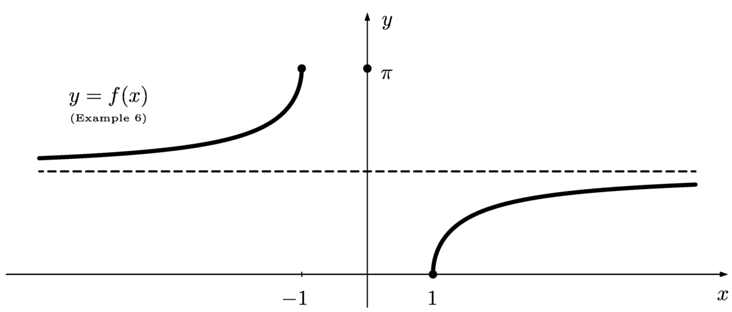

Fig. 8. The graph of function from Example 6

Example 7. Determine types of discontinuities of function

$$f(x) = \begin{cases} \arccos \frac{1}{x} & \text{if} \quad x \notin \{-1, 1\} \\ 2 + \arccos x & \text{if} \quad x = 1. \end{cases}$$

We start with the domain of function f. After calculations similar to these in the previous example, we get $D_f = (-\infty, -1) \cup [1, +\infty)$.

Note that -1 is the cluster point of D_f which does not belong to D_f so f has the discontinuity at -1. We have

$$\lim_{x \to -1} f(x) = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \arccos \frac{1}{x} = \arccos(-1) = \pi.$$

Thus, f has the removable discontinuity (I type) at -1.

The function may have (but need not) a discontinuity at 1. We have

$$f(1) = 2 + \arccos 1 = 2 + 0 = 2,$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \arccos \frac{1}{x} = \arccos 1 = 0.$$

Hence, f has the removable discontinuity (I type) at 1. The graph of this function is presented in Fig. 9.

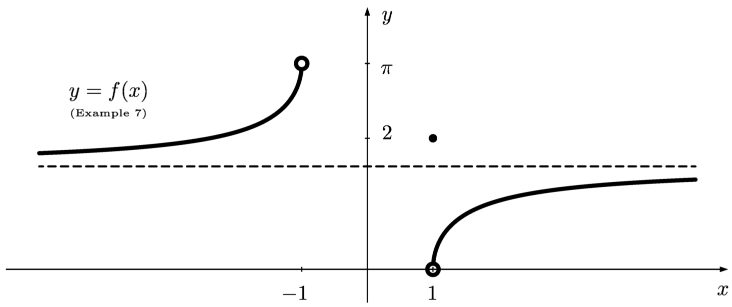

Fig. 9. The graph of function from Example 7

3. An exercise to be solved by oneself

Find discontinuities of functions listed below:

a) $f(x) = \dfrac{\sin 2x}{\sin 4x}$

b) $f(x) = \begin{cases} \arctan \frac{x}{x-2} & \text{if} \quad x < 2 \\ \frac{\sqrt{x^2+9}-5}{x-4} & \text{if} \quad x \geq 2 \end{cases}$

c) $f(x) = \begin{cases} \ln(9-x^2) & \text{if} \quad x \neq 5 \\ \ln 3 & \text{if} \quad x = 5 \end{cases}$

d) $f(x) = \begin{cases} \frac{e^x-1}{x^2-2x} & \text{if} \quad x < 1 \\ \ln x - e^x + 1 & \text{if} \quad x \geq 1 \end{cases}$

e) $f(x) = \begin{cases} \arccos \frac{1}{x} & \text{if} \quad x \leq -2 \\ 2\text{arccot}\,(x+2) & \text{if} \quad x > 2 \end{cases}$

f) $f(x) = \sqrt{x^2-4} + \sqrt{4-x^2}$

Answers:

a) function f has removable discontinuities at points $k\pi$ and $k\pi + \frac{\pi}{2}$ $(l \in \mathbb{Z})$;
function f has infinite jumps at points $k\pi + \frac{\pi}{4}$ and $k\pi + \frac{3\pi}{2}$ $(l \in \mathbb{Z})$;

b) function f has finite jump at 2 and removable discontinuity at 4;

c) function f has infinite jumps at 3 and -3;

d) function f has removable discontinuity at 0;

e) function f does not have any discontinuities;

f) function f does not have any discontinuities (note that its domain is $D = \{-2, 2\}$).

References

1. E. Łobos, B. Sikora, *Calculus and differential equations in exercises*, Gliwice 2012, p. 27.
2. B. Sikora, E. Łobos, *A first course in calculus*, Gliwice 2007, pp. 141-143.