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Determinants - a short tutorial

Abstract. This short course is intended for beginners and, therefore, it contains only basic information on the determinant. A recursive definition of the determinant is given based on the Laplace expansion. It is shown how to compute determinants of different orders on the basis of the definition, using Laplace's formula against any row or column, and by means of elementary operations. Basic theorems are given to facilitate the calculation of determinants. The content is illustrated with numerous solved examples, with self-testing examples and answers provided at the end of the course.
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1. Introduction

In linear algebra, an important concept related to matrices is the determinant. The ability to calculate determinants is needed not only in mathematics, but it is also useful for engineers in various fields.

Basic information about matrices and operations on them can be found in the article [1], while more detailed information is provided in algebra textbooks such as [2] or [3].

In this paper we will denote matrices by bold capital letters and their entries by lowercase letters as

$$\mathbf{A} = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ – a square matrix of order } n.$$

In the examples below, the elements of the matrix are real numbers, but all the definitions, theorems and considerations apply to complex numbers as well.

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2. Defining the determinant of a matrix and calculating determinants from definitions

The determinant of a matrix can be defined in several ways, i.e. – axiomatically, recursively or by permutations. Although there are various formal definitions of the determinant of a matrix, they all denote the same thing, which is – the number assigned to a square matrix (the determinant is thus a function).

Let us quote a recursive definition.

Definition 1. ***The determinant*** *of a square matrix* $\mathbf{A}$ *is the function that assigns a real number* $\det(\mathbf{A})$ *to each square real matrix as follows:*

1. *If* $\mathbf{A} = [a_{11}]$ *is of order* $n = 1$*, then* $\det(\mathbf{A}) = a_{11}$.
2. *If* $\mathbf{A} = [a_{ij}]$ *is of order* $n \geqslant 2$*, then:*

$$\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + \ldots a_{1n}A_{1n} = \sum_{j=1}^{n} a_{1j}A_{1j}, \tag{1}$$

where A_{ij} *is the* ***cofactor*** *of the element* a_{ij}*, i.e. the product of* $(-1)^{i+j}$ *and the determinant of the matrix obtained from* $\mathbf{A}$ *by removing (crossing out) the* i*-th row and* j*-th column.*

The concept of a **cofactor** appears in the definition above. Let us look at a specific matrix as an example.

Example 1. Let us find the cofactors for selected elements of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -2 \\ 3 & -3 & 4 & -4 \\ 5 & -5 & 6 & -6 \\ 7 & -7 & 8 & -8 \end{bmatrix}$.

The cofactor of the element a_{11}, i.e.

$$A_{11} = (-1)^2 \cdot \begin{vmatrix} -3 & 4 & -4 \\ -5 & 6 & -6 \\ -7 & 8 & -8 \end{vmatrix}$$

is equal to the product (-1) raised to the power of the sum of the row and column indices $(1 + 1 = 2)$ multiplied by the determinant of the matrix formed from $\mathbf{A}$ by removing the first row and the first column $\begin{bmatrix} -|- & -- & -- & -- \\ | & -3 & 4 & -4 \\ | & -5 & 6 & -6 \\ | & -7 & 8 & -8 \end{bmatrix}$.

The cofactor of element a_{14}, i.e.

$$A_{14} = (-1)^5 \cdot \begin{vmatrix} 3 & -3 & 4 \\ 5 & -5 & 6 \\ 7 & -7 & 8 \end{vmatrix}$$

is equal to $(-1)^{1+4}$ multiplied by the determinant of the matrix obtained from **A** by removing the first row and the fourth column $\begin{bmatrix} -- & -- & -- & -|- \\ 3 & -3 & 4 & | \\ 5 & -5 & 6 & | \\ 7 & -7 & 8 & | \end{bmatrix}$.

Similarly, we can determine the cofactors of the other elements in the first row a_{12}, a_{13}.

$$A_{12} = -\begin{vmatrix} 3 & 4 & -4 \\ 5 & 6 & -6 \\ 7 & 8 & -8 \end{vmatrix}, \qquad A_{13} = -\begin{vmatrix} 3 & -3 & -4 \\ 5 & -5 & -6 \\ 7 & -7 & -8 \end{vmatrix}.$$

Although only the cofactors of the elements in the first row are used in the above definition, it is sometimes necessary to calculate them[1] for the other entries as well.

For example the cofactor of the element a_{23}, i.e. $A_{23} = (-1)^5 \cdot \begin{vmatrix} 3 & -3 & -4 \\ 5 & -5 & -6 \\ 7 & -7 & -8 \end{vmatrix}$ is equal to the product (-1) raised to the power of the sum of the row and column indices $2 + 3$ multiplied by the determinant of the matrix obtained from **A** by removing the second row and the third column

$$\begin{bmatrix} 1 & -1 & | & -2 \\ -- & -- & -|- & -- \\ 5 & -5 & | & -6 \\ 7 & -7 & | & -8 \end{bmatrix}.$$

Note that if we want to calculate the determinant of a matrix of order n it is not enough to use the formula (1). If we decompose it from the cofactors, we get new determinants for matrices of order $n - 1$.

In the next chapter we will show how to use the given definition to denote the value of the determinant.

Since the determinant is a function denoted by **det**, the fact that it assigns one real number to each square matrix with real elements can be written in the following way:

$$\det : M_n(\mathbb{R}) \rightarrowtail \mathbb{R},$$

where $M_n(\mathbb{R})$ is the set of all square matrices with elements from the set $\mathbb{R}$ of real numbers.

We use the terms:

$$\det(\mathbf{A}) \text{ or } |\mathbf{A}|, \quad \det \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 0 \\ \frac{1}{3} & \pi & 0.32 \end{bmatrix} \text{ or } \begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & 0 \\ \frac{1}{3} & \pi & 0.32 \end{vmatrix} \quad \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \text{ or } \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

[1] For example, to compute inverse matrices.

Note on the designations

Please note that when we use the name of the function, i.e. the symbol det, the matrix is written with square brackets (standard as a matrix). The simplified notation (without the function name) is just dashes around the elements of the matrix (the square brackets disappear).

The det function (because it is a function) assigns exactly one number to each square matrix – each matrix has one determinant. However, it is not an injective function. Different matrices can be assigned the same number, i.e. different matrices can have the same (equal) determinants.

2.1. Determinants of matrices of order one

The calculation of the determinant of a matrix of order one[2] follows directly from point 1 of the definition 1:

$$\det([a]) = a \text{ for all } a \in \mathbb{R}.$$

For example, for $\mathbf{A} = [3]$ $\det(\mathbf{A}) = 3$ and for $\mathbf{B} = [-\frac{2\pi}{7}]$ $\det(\mathbf{B}) = -\frac{2\pi}{7}$.

2.2. Determinants of matrices of order two

Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order two.

To calculate its determinant using the definition, we need to find the cofactors of the successive elements standing in the first row of the matrix and then use the formula (1).
The cofactors of the elements are:

- for a_{11} it is $A_{11} = (-1)^{1+1} \det([a_{22}]) = a_{22}$,
- for a_{12} it is $A_{12} = (-1)^{1+2} \det([a_{21}]) = -a_{21}$,

Consequently according to the definition and equation (1) we have

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} = \mathbf{a_{11}a_{22} - a_{12}a_{21}}.$$

To make it easier to remember, we can visualise this as:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \mathbf{a_{11}a_{22}} - \mathbf{a_{12}a_{21}}. \tag{2}$$

[2] For matrices of order one, we discourage the use of straight dashes to denote the determinant, as this denotation may be confused with the absolute value.

Example 2. Calculate the determinant of matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

By the definition we have

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 \cdot (-1)^{1+1} \det([1]) + 3 \cdot (-1)^{1+2} \det([4]) = 2 \cdot 1 - 3 \cdot 4 = -10.$$

We can use the formula (2) to do it faster

$$\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = \mathbf{2 \cdot 1 - 3 \cdot 4 = -10}.$$

2.3. Determinants of matrices of order three

According to the definition (1)

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

Let us start by calculating the cofactors A_{11}, A_{12}, A_{13} for the elements of the first row.

To determine A_{11} we multiply $(-1)^{1+1}$ by the determinant of the matrix left after removing the first row and the first column from $\mathbf{A}$ $\begin{bmatrix} -|- & -- & -- \\ | & a_{22} & a_{23} \\ | & a_{32} & a_{33} \end{bmatrix}$.

Thus $A_{11} = (-1)^2 \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$.

To determine A_{12} we multiply $(-1)^{1+\mathbf{2}}$ by the determinant of the matrix that remained when we removed the first row and the second column from $\mathbf{A}$ $\begin{bmatrix} -- & -|- & -- \\ a_{21} & | & a_{23} \\ a_{31} & | & a_{33} \end{bmatrix}$.

Thus $A_{12} = (-1)^3 \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -a_{21}a_{33} + a_{23}a_{31}$.

Similarly, by deleting the first row and the third column $\begin{bmatrix} -- & -- & -|- \\ a_{21} & a_{22} & | \\ a_{31} & a_{32} & | \end{bmatrix}$ we get

$A_{13} = (-1)^{1+\mathbf{3}} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (-1)^4(a_{21}a_{32} - a_{22}a_{31})$.

Finally we obtain:

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} =$$

$$= a_{11}(-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^4 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Using the formula (2) for matrices of order two we have

$$\det(\mathbf{A}) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}),$$

and after removing the parentheses and rearranging

$$\det(\mathbf{A}) = \mathbf{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})}. \tag{3}$$

Example 3. Calculate the determinant of the matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ 2 & 1 & 0 \end{bmatrix}$ using the definition given.

By expanding the determinant of a given matrix with respect to the first row, as defined, we obtain:

$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ 2 & 1 & 0 \end{vmatrix} = b_{11}B_{11} + b_{12}B_{12} + b_{13}B_{13} =$$

$$= 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} + 2 \cdot (-1)^{1+2} \begin{vmatrix} 4 & -2 \\ 2 & 0 \end{vmatrix} + 3 \cdot (-1)^{1+3} \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}.$$

We need to calculate the three determinants of order two. We can use the quick formula (2)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ 2 & 1 & 0 \end{vmatrix} = 1 \cdot (1 \cdot 0 - (-2) \cdot 1) + 2 \cdot (-1)(4 \cdot 0 - (-2) \cdot 2) + 3 \cdot (4 \cdot 1 - 1 \cdot 2) = 0.$$

We could directly use the formula (3) directly and get the same result

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ 2 & 1 & 0 \end{vmatrix} = 1 \cdot 1 \cdot 0 + 2 \cdot (-2) \cdot 2 + 3 \cdot 4 \cdot 1 - (1 \cdot (-2) \cdot 1 + 2 \cdot 4 \cdot 0 + 3 \cdot 1 \cdot 2) =$$

$$= 0 - 8 + 12 - (-2 + 0 + 6) = 0.$$

Although the formula in the form (3) is not easy to remember, we can add the first two columns to the matrix and use the **Sarrus Rule** as shown below:

$$\begin{array}{ccc|cc} + & + & + & & \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22}1 & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ - & - & - & & \end{array} =$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}).$$

The application of the Sarrus Rule to exercise 2 is as follows:

$$\begin{array}{ccc|cc} + & + & + & & \\ 1 & 2 & 3 & 1 & 2 \\ 4 & 1 & -2 & 4 & 1 \\ 2 & 1 & 0 & 2 & 1 \\ - & - & - & & \end{array} =$$

$$= (1 \cdot 1 \cdot 0 + 2 \cdot (-2) \cdot 2 + 3 \cdot 4 \cdot 1) - (1 \cdot (-2) \cdot 1 + 2 \cdot 4 \cdot 0 + 3 \cdot 1 \cdot 2) = 0.$$

Notice

It is very important to note that this type of rule (with column addition) applies to matrices of order three only. There is no rule analogous to the Sarrus Rule for matrices of higher orders.

2.4. Fourth-order or higher order matrix determinants

There is no point in deriving a formula analogous to (2) or (3) for matrices of order four or greater, as these would be very long and unsuitable for memorisation. We will show by example how to deal with the application of definitions to such matrices.

Example 4. Using the definition, calculate the determinant of the matrix $\mathbf{C} = \begin{bmatrix} -1 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 & 1 \end{bmatrix}$.

$$\det(\mathbf{C}) = c_{11}C_{11} + c_{12}C_{12} + c_{13}C_{13} + c_{14}C_{14} + c_{15}C_{15}.$$

Since the elements c_{12}, c_{13}, a_{14} are equal to 0 it is sufficient to calculate $c_{11}C_{11} + c_{15}C_{15}$.

$$\det(\mathbf{C}) = c_{11}c_{11} + c_{15}C_{15} = -1 \cdot (-1)^2 \begin{vmatrix} 1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & -2 & 0 \\ 2 & -1 & 0 & 1 \end{vmatrix} + 2 \cdot (-1)^6 \begin{vmatrix} 2 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 2 & -1 & 0 \end{vmatrix}.$$

Now we need to calculate the two determinants of order four. Taking into account the zero elements we have:

$$\det(\mathbf{C}) = -1\left(1\cdot(-1)^2\begin{vmatrix}1&1&1\\0&-2&0\\-1&0&1\end{vmatrix}+(-2)\cdot(-1)^4\begin{vmatrix}1&1&1\\-2&0&0\\2&-1&1\end{vmatrix}\right)+$$
$$+2\left(2\cdot(-1)^2\begin{vmatrix}1&1&1\\-2&0&-2\\2&-1&0\end{vmatrix}+1\cdot(-1)^3\begin{vmatrix}1&1&1\\0&0&-2\\0&-1&0\end{vmatrix}-2\cdot(-1)^5\begin{vmatrix}1&1&1\\0&-2&0\\0&2&-1\end{vmatrix}\right).$$

To calculate the resulting determinants of order three we can use definition 1 or the Sarrus Rule as shown above.

$$\begin{vmatrix}1&1&1\\0&-2&0\\-1&0&1\end{vmatrix}=1\cdot(-1)^2\begin{vmatrix}-2&0\\0&1\end{vmatrix}+1\cdot(-1)^3\begin{vmatrix}0&0\\-1&1\end{vmatrix}+1\cdot(-1)^4\begin{vmatrix}0&-2\\-1&0\end{vmatrix}=-2+0-2=-4,$$

$$\begin{vmatrix}1&1&1\\-2&0&0\\2&-1&1\end{vmatrix}=1\cdot(-1)^2\begin{vmatrix}0&0\\-1&1\end{vmatrix}+1\cdot(-1)^3\begin{vmatrix}-2&0\\2&1\end{vmatrix}+1\cdot(-1)^4\begin{vmatrix}-2&0\\2&-1\end{vmatrix}=0+2+2=4,$$

$$\begin{array}{ccc}+&+&+\end{array}$$
$$\left|\begin{array}{ccc|cc}1&1&1&1&1\\0&-2&0&0&-2\\-1&0&1&-1&0\end{array}\right. = -2+0+0-(2+0+0)=-4,$$
$$\begin{array}{ccc}-&-&-\end{array}$$

$$\left|\begin{array}{ccc|cc}1&1&1&1&1\\0&0&-2&0&0\\0&-1&0&0&-1\end{array}\right. = 0+0+0-(0+2+0)=-2,$$

$$\left|\begin{array}{ccc|cc}1&1&1&1&1\\0&-2&0&0&-2\\0&2&-1&0&2\end{array}\right. = 2+0+0-(0+0+0)=2.$$

]

The final result is:

$$\det(\mathbf{C}) = -1(-4-2\cdot4)+2(2(-4)-1(-2)+2\cdot2)=8.$$

To find the determinant of a given matrix, we had to perform quite a lot of calculations. However, several elements were zero, which reduced the number of calculations needed. If no element of the matrix **C** was zero, then five determinants of order four would have to be calculated, which would give twenty determinants of order three, i.e. a very large number of calculations. Therefore, this method (based on the definition) is rarely used for higher order matrices.

Alternative methods are presented below.

3. Laplace expansion

When calculating the determinant of a matrix according to the definition 1, the determinant to be calculated[3] is replaced by the sum of several lower-order determinants. The formula (1) is called the expansion of the determinant with respect to the first row. A similar technique can be applied to any row or column of the matrix. It is called the Laplace expansion.

The Laplace expansion with respect to the i-th row, for a matrix of order $n \geq 2$ is presented in the formula:

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{ik} A_{ik} = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \tag{4}$$

where A_{ik} is the cofactor of the element a_{ik} defined in definition 1.

The Laplace expansion with respect to the j-th column is presented in the formula:

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{kj} A_{kj} = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}. \tag{5}$$

The number computed by the Laplace expansion with respect to any row (or column) is equal to the determinant computed by the definition.

To illustrate this technique, let us calculate the determinant of the matrix given in Example 2.

Example 5. Calculate the determinant of the matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ 2 & 1 & 0 \end{bmatrix}$ using the Laplace expansion with respect to the third row.

We will use the equation (4) with the labels B_{31}, B_{32}, B_{33} as cofactors for the respective elements b_{31}, b_{32}, b_{33}.

$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \end{vmatrix} = b_{31}B_{31} + b_{32}B_{32} + b_{33}B_{33} =$$

$$= 2 \cdot (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} + 1 \cdot (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} + 0 \cdot (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} =$$

$$= 2 \cdot (-4 - 3) + 1 \cdot (-1)(-2 - 12) + 0 = 0.$$

Example 6. Calculate the determinant of the matrix $\mathbf{D} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ using the Laplace expansion with respect to a column of your choice.

[3] for matrices of order higher than two

For the Laplace expansion, it is best to choose the column with the most zero elements. In this case, it is the fourth column. So let us apply the formula (5) to the fourth column:

$$\det(\mathbf{D}) = \begin{vmatrix} 2 & 2 & 2 & \mathbf{2} \\ 3 & 1 & 2 & \mathbf{0} \\ 2 & 1 & 0 & \mathbf{0} \\ 0 & -1 & 0 & \mathbf{0} \end{vmatrix} = 2\cdot(-1)^{1+4} \begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix}.$$

We can apply the Laplace expansion again to the last (third) column

$$\det(\mathbf{D}) = -2 \begin{vmatrix} 3 & 1 & \mathbf{2} \\ 2 & 1 & \mathbf{0} \\ 0 & -1 & \mathbf{0} \end{vmatrix} = -2\cdot 2\cdot(-1)^4 \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = -4(-2-0) = 8.$$

Example 7. Calculate the determinant of the matrix $\mathbf{F} = \begin{bmatrix} 2 & 0 & 1 & -3 & 7 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$.

In order to calculate this determinant, we are going to apply the Laplace expansion several times with respect to the first column.

$$\det(\mathbf{F}) = \begin{vmatrix} \mathbf{2} & 0 & 1 & -3 & 7 \\ \mathbf{0} & -1 & 3 & 5 & 0 \\ \mathbf{0} & 0 & 3 & 2 & 1 \\ \mathbf{0} & 0 & 0 & 7 & -2 \\ \mathbf{0} & 0 & 0 & 0 & 4 \end{vmatrix} = 2\cdot(-1)^2 \begin{vmatrix} \mathbf{-1} & 3 & 5 & 0 \\ \mathbf{0} & 3 & 2 & 1 \\ \mathbf{0} & 0 & 7 & -2 \\ \mathbf{0} & 0 & 0 & 4 \end{vmatrix} = 2\cdot(-1)\cdot(-1)^2 \begin{vmatrix} \mathbf{3} & 2 & 1 \\ \mathbf{0} & 7 & -2 \\ \mathbf{0} & 0 & 4 \end{vmatrix} =$$

$$= 2\cdot(-1)\cdot 3\cdot(-1)^2 \begin{vmatrix} 7 & -2 \\ 0 & 4 \end{vmatrix} = 2\cdot(-1)\cdot 3\cdot 7\cdot 4 = -168.$$

The matrix $\mathbf{F}$ is called a triangular (upper) matrix because of its characteristic appearance – all entries below the main diagonal (marked in blue) are zero (marked in red):

$$\mathbf{F} = \begin{bmatrix} \color{blue}{2} & 0 & 1 & -3 & 7 \\ \color{red}{0} & \color{blue}{-1} & 3 & 5 & 0 \\ \color{red}{0} & \color{red}{0} & \color{blue}{3} & 2 & 1 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{blue}{7} & -2 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{blue}{4} \end{bmatrix}$$

As you can easily see, the determinant of this matrix is the product of the elements on the diagonal:

$$\det(\mathbf{F}) = \begin{vmatrix} \color{blue}{2} & 0 & 1 & -3 & 7 \\ 0 & \color{blue}{-1} & 3 & 5 & 0 \\ 0 & 0 & \color{blue}{3} & 2 & 1 \\ 0 & 0 & 0 & \color{blue}{7} & -2 \\ 0 & 0 & 0 & 0 & \color{blue}{4} \end{vmatrix} = \color{blue}{2\cdot(-1)\cdot 3\cdot 7\cdot 4}.$$

Similarly, the matrix $\begin{bmatrix} a & 0 & 0 \\ -2 & b & 0 \\ 1 & 0 & c \end{bmatrix}$ is a triangular (lower) matrix and its determinant is equal to abc.

The Reader can check this by calculating it using one of the methods described.

Note: the matrices $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -2 \\ -2 & 1 & 4 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 0 & 0 \\ 9 & 0 & 0 & 0 \end{bmatrix}$ **are not** triangular matrices!

4. Properties of determinants

Let's go through some properties of determinants that can make their calculation much easier. Some of the most useful are listed below.

Theorem 1 (reflection property). *The determinant of a matrix transposed*[4] *into a square matrix* $\mathbf{A}$ *is the same as the determinant of the matrix* $\mathbf{A}$*:*

$$\det(\mathbf{A}^T) = \det(\mathbf{A}).$$

For instance $\begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -5.$

Theorem 2 (all-zero property). *If all the elements in any row of a square matrix are zeros, then the determinant of that matrix is zero.*

For example $\begin{vmatrix} \frac{2}{3} & 3 & \pi & \ln 7 \\ 0 & 0 & 0 & 0 \\ -27 & 3 & \frac{7}{5} & 13 \\ 2 & \sin 3 & 4 & 0 \end{vmatrix} = 0.$

Theorem 3 (proportionality or repetition property). *If, in a square matrix, the elements of two rows are proportional to each other or two rows are the same, the determinant of such a matrix is equal to zero.*

This is the case in the given matrix, where the second and fourth rows are proportional (the constant of proportionality is -2):

$$\begin{vmatrix} 10 & \frac{1}{17} & 1 & -3 \\ \cos 2 & -3 & \pi & 0 \\ -0.1 & 3 & -1 & e^2 \\ -2\cos 2 & 6 & -2\pi & 0 \end{vmatrix} = 0.$$

[4] What is a transposed matrix the Reader can see in the [1].

Theorem 4 (switching property). *If we swap one row with another row in a given matrix* $\mathbf{A}$, *we get a matrix whose determinant is a number opposite to the determinant of matrix* $\mathbf{A}$.

Knowing that $\det(\mathbf{A}) = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -15$ we can, use the switching property to determine that

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 4 & 3 & 2 \end{vmatrix} \overset{\text{row } 1 \leftrightarrow \text{row } 3}{=} -\begin{vmatrix} 4 & 3 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 15.$$

Theorem 5 (scalar multiple property). *If, in a given matrix* $\mathbf{A}$, *we multiply all the elements of a certain row by the number* k $(k \in \mathbb{R} \setminus 0\})$, *we obtain a matrix whose determinant is equal to* $k\det(\mathbf{A})$.

That is, if $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ -6 & 2 & 3 \end{vmatrix} = 16$ then

$$\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 1 \\ -6 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2\cdot 1 & -1 & 0 \\ 2\cdot 2 & 3 & 1 \\ 2\cdot(-3) & 2 & 3 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 32.$$

From the above property we can see how the determinant is multiplied by a number. Let's look at an example of how this works:

$$\frac{1}{2}\cdot\begin{vmatrix} 2 & -4 & 0 \\ 4 & 2 & 1 \\ -6 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 0 \\ 2 & 2 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

but also

$$\frac{1}{2}\cdot\begin{vmatrix} 2 & -4 & 0 \\ 4 & 2 & 1 \\ -6 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 \\ 4 & 2 & 1 \\ -6 & 2 & 3 \end{vmatrix} \quad \text{and} \quad \frac{1}{2}\cdot\begin{vmatrix} 2 & -4 & 0 \\ 4 & 2 & 1 \\ -6 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -4 & 0 \\ 2 & 1 & \frac{1}{2} \\ -6 & 2 & 3 \end{vmatrix}.$$

Theorem 6 (triangle property). *If every entry of a matrix below (or above) the main diagonal is zero, (i.e. it is a triangular matrix), then the determinant of that matrix is equal to the product of the diagonal expressions.*

In that case: $\begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} = -2$, and $\begin{vmatrix} 2 & 0 & 0 \\ -1 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix} = 6$, and $\begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 24.$

Another illustration of this property is in Exercise 6, which we did earlier.

Theorem 7 (invariance property). *If, in a given matrix* $\mathbf{A}$, *we multiply the elements of a certain row by the number* k *and add them to the corresponding elements of another row, we obtain a matrix whose determinant is the same as* $\det(\mathbf{A})$.

Using this theorem, we can perform the following transformations that do not change the value of the determinant

$$\det\begin{bmatrix}1&-1&1&-1&1\\2&-1&1&-1&3\\0&0&1&-1&1\\-3&3&-3&4&-4\\0&0&0&0&1\end{bmatrix}\overset{r_1\cdot(-2)+r_2}{=}\begin{vmatrix}1&-1&1&-1&1\\0&1&-1&1&1\\0&0&1&-1&1\\-3&3&-3&4&-4\\0&0&0&0&1\end{vmatrix}\overset{3r_1+r_4}{=}\begin{vmatrix}1&-1&1&-1&1\\0&1&-1&1&1\\0&0&1&-1&1\\0&0&0&1&-1\\0&0&0&0&1\end{vmatrix}$$

to obtain the determinant of the triangular matrix. The first row (r_1) was multiplied by -2 and added to the second row (r_2) and then the first row multiplied by 3 ($3r_1$) was added to the fourth row (r_4).

According to the previous theorem, the last determinant of a triangular matrix is equal to the product of the elements on the main diagonal $\begin{vmatrix}1&-1&1&-1&1\\0&1&-1&1&1\\0&0&1&-1&1\\0&0&0&1&-1\\0&0&0&0&1\end{vmatrix}=1.$

As a direct consequence of Theorem 1, an important fact is given in the box below.

Relevant fact

Although formulated for the rows, the Theorems 2 3,4,5 and 7 are also valid when the operations described are performed on the columns of the matrix.

4.1. Calculate determinants using elementary operations.

ERO – the elementary row operations described in [1], and their counterparts on the columns – ECO, can be very useful in computing determinants. We have three such operations:

ERO1 (ECO1) – swapping rows (columns). According to the switching property (Theorem 4), this operation changes the value of the determinant of the matrix to the opposite number.

ERO2 (ECO2) – multiplication of rows (columns). If we multiply all elements of a row (column) by a non-zero number k, then according to the property of scalar multiplication (Theorem 5) the determinant is also multiplied by k.

ERO3 (ECO3) – addition of rows (columns). As is clear from the invariance property (Theorem 7), performing this operation does not change the value of the determinant.

We can use the operations described in a number of ways. Using one or more of the EROs (ECOs), we can transform a given matrix into one that has many zeros in a particular row (or column) and then perform the Laplace expansion against that row (column). We can also transform a matrix $\mathbf{A}$, keeping track of how its determinant changes, into a triangular matrix and (according to Theorem 6) multiply the entries on its main diagonal

Example 8. Calculate the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 & 0 \\ 2 & 1 & -3 & 1 \\ 4 & 2 & -4 & 3 \\ 0 & 1 & 2 & -2 \end{bmatrix}$.

We will start the calculation by applying elementary operations to generate zeros in the selected row or column. Obviously, it is most convenient to create additional zeros in the row or column where some already exist. Here, we will use the fourth column.

Since the element $a_{21} = 1$, it will be easy to get additional zeros by applying the second row to the third and fourth rows. More precisely, we will use ERO3 twice:

1) we will multiply the second row by -3 and add it to third row,

2) then after multiplying row two by 2 we will add it to row for.

Therefore:

$$\det \mathbf{A} = \begin{vmatrix} 3 & 2 & -1 & 0 \\ 2 & 1 & -3 & 1 \\ 4 & 2 & -4 & 3 \\ 0 & 1 & 2 & -2 \end{vmatrix} \overset{(-3)r_2 + r_3}{=} \begin{vmatrix} 3 & 2 & -1 & 0 \\ 2 & 1 & -3 & 1 \\ -2 & -1 & 5 & 0 \\ 0 & 1 & 2 & -2 \end{vmatrix} \overset{2r_2 + r_4}{=} \begin{vmatrix} 3 & 2 & -1 & 0 \\ 2 & 1 & -3 & 1 \\ -2 & -1 & 5 & 0 \\ 4 & 3 & -4 & 0 \end{vmatrix}.$$

Notice

It is important to note that by using ERO3 and acting on the third (or fourth) row with the second row, we are changing the elements in the third (fourth) row only, leaving the second row unchanged.

Now we can apply the Laplace's expansion to the fourth column, as follows

$$\begin{vmatrix} 3 & 2 & -1 & 0 \\ 2 & 1 & -3 & 1 \\ -2 & -1 & 5 & 0 \\ 4 & 3 & -4 & 0 \end{vmatrix} = (-1)^{2+4} \begin{vmatrix} 3 & 2 & -1 \\ -2 & -1 & 5 \\ 4 & 3 & -4 \end{vmatrix}.$$

And generate zeros again. For example, use the second row.

$$\begin{vmatrix} 3 & 2 & -1 \\ -2 & -1 & 5 \\ 4 & 3 & -4 \end{vmatrix} \overset{2r_2 + r_1}{=} \begin{vmatrix} -1 & 0 & -11 \\ -2 & -1 & 5 \\ 4 & 3 & -4 \end{vmatrix} \overset{3r_2 + r_3}{=} \begin{vmatrix} -1 & 0 & -11 \\ -2 & -1 & 5 \\ -2 & 0 & 11 \end{vmatrix}.$$

After applying the Laplace expansion to the second row, we obtain a determinant of order two, which is easy to calculate.

$$\begin{vmatrix} -1 & 0 & -11 \\ -2 & -1 & 5 \\ -2 & 0 & 11 \end{vmatrix} = (-1)^4 \begin{vmatrix} -1 & -11 \\ -2 & 11 \end{vmatrix} = -11 - (-2)(-11) = -33.$$

5. Solved examples

Example 9. Calculate the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ -2 & 0 & -1 \\ -1 & 2 & 8 \end{bmatrix}$ by transforming it into a triangular matrix.

We start by swapping the first and third rows to get the element $a_{11} = -1$. This will make further calculations easier. Remember to change the sign of the determinant according to Theorem 4.

$$\det \mathbf{A} = \begin{vmatrix} 3 & 2 & 4 \\ -2 & 0 & -1 \\ -1 & 2 & 8 \end{vmatrix} \overset{r_1 \leftrightarrow r_3}{=} - \begin{vmatrix} -1 & 2 & 8 \\ -2 & 0 & -1 \\ 3 & 2 & 4 \end{vmatrix}.$$

For convenience we can use the Theorem 5 and exclude the factor -1 from the second row before the determinant. This will make the '-' sign before the determinant disappear.

$$\det \mathbf{A} = -(-1) \begin{vmatrix} -1 & 2 & 8 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 8 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}.$$

We now carry out some more elementary operations to produce a triangular matrix, whose determinant we calculate using Theorem 6.

$$\begin{vmatrix} -1 & 2 & 8 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} \begin{matrix} 2r_1 + r_2 \\ = \\ 3r_1 + r_3 \end{matrix} \begin{vmatrix} -1 & 2 & 8 \\ 0 & 4 & 17 \\ 0 & 8 & 28 \end{vmatrix} \overset{-2r_2 + r_3}{=} \begin{vmatrix} -1 & 2 & 8 \\ 0 & 4 & 17 \\ 0 & 0 & 6 \end{vmatrix} = -24.$$

Example 10. Calculate $\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & -2 \end{vmatrix}$.

If the content of the task does not dictate a method (as was the case in Exercise 8), we can choose any approach. To calculate the third-order determinants, students are most likely to use Sarrus Rule. Teaching experience shows that they often make mistakes in their calculations. It is often quicker and easier to use elementary operations and the Laplace expansion.

$$\begin{vmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & -2 \end{vmatrix} \begin{matrix} 2r_2 + r_1 \\ = \\ r_2 + r_3 \end{matrix} \begin{vmatrix} 8 & 0 & 7 \\ 3 & 1 & 2 \\ 7 & 0 & 0 \end{vmatrix} \overset{\text{Laplace to } r_3}{=} 7 \cdot (-1)^{3+1} \begin{vmatrix} 0 & 7 \\ 1 & 2 \end{vmatrix} = 7(-7) = -49.$$

The Reader is free to use the Sarrus rule for themselves and see if they come up with teh same result.

Example 11. Calculate the determinant $\begin{vmatrix} 2 & -4 & 6 & 2 & -2 \\ 2 & -3 & 8 & -1 & 1 \\ -1 & 7 & 2 & -7 & 0 \\ 3 & -8 & -7 & 10 & -1 \\ 2 & -1 & -5 & 3 & -1 \end{vmatrix}$.

First, note that all the elements in the first row are multiples of 2, so we can use the Theorem 5 and get:

$$\begin{vmatrix} 2 & -4 & 6 & 2 & -2 \\ 2 & -3 & 8 & -1 & 1 \\ -1 & 7 & 2 & -7 & 0 \\ 3 & -8 & -7 & 10 & -1 \\ 2 & -1 & -5 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 2 & -3 & 8 & -1 & 1 \\ -1 & 7 & 2 & -7 & 0 \\ 3 & -8 & -7 & 10 & -1 \\ 2 & -1 & -5 & 3 & -1 \end{vmatrix}$$

We can now use the first row and, using ERO3, generate zeros in the first column.

$$\begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 2 & -3 & 8 & -1 & 1 \\ -1 & 7 & 2 & -7 & 0 \\ 3 & -8 & -7 & 10 & -1 \\ 2 & -1 & -5 & 3 & -1 \end{vmatrix} \begin{matrix} -2r_1 + r_2 \\ = \\ r_1 + r_3 \end{matrix} \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 5 & -6 & 5 & -1 \\ 3 & -8 & -7 & 10 & -1 \\ 2 & -1 & -5 & 3 & -1 \end{vmatrix} \begin{matrix} -3r_1 + r_4 \\ = \\ -2r_1 + r_5 \end{matrix} \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 5 & -6 & 5 & -1 \\ 0 & -2 & 7 & -16 & 2 \\ 0 & 3 & 1 & -11 & 1 \end{vmatrix}.$$

The next step is to use the second row to set the elements in the second column to zero. Thanks to the fact that the element at the beginning of the second row is now zero, we will not lose the zeros we have previously made in the first column.

$$\begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 5 & -6 & 5 & -1 \\ 0 & -2 & 7 & -16 & 2 \\ 0 & 3 & 1 & -11 & 1 \end{vmatrix} \begin{matrix} -5r_2 + r_3 \\ = \\ 2r_2 + r_4 \\ -3r_2 + r_5 \end{matrix} \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 9 & -5 & -16 \\ 0 & 0 & 1 & -12 & 8 \\ 0 & 0 & 10 & -17 & -8 \end{vmatrix}.$$

Swap rows three and four and then contunue using the third row to zero the elements in the third column.

$$\begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 9 & -5 & -16 \\ 0 & 0 & 1 & -12 & 8 \\ 0 & 0 & 10 & -17 & -8 \end{vmatrix} \begin{matrix} r_3 \leftrightarrow r_4 \\ = \end{matrix} - \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 1 & -12 & 8 \\ 0 & 0 & 9 & -5 & -16 \\ 0 & 0 & 10 & -17 & -8 \end{vmatrix} \begin{matrix} -9r_3 + r_4 \\ = \\ -10r_3 + r_5 \end{matrix}$$

$$= - \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 1 & -12 & 8 \\ 0 & 0 & 0 & 103 & -88 \\ 0 & 0 & 0 & 103 & -88 \end{vmatrix}$$

Since the two rows r_4 and r_5 are the same, we know (by the Theorem 3) without further calculation that the determinant is equal to 0.

$$\begin{vmatrix} 2 & -4 & 6 & 2 & -2 \\ 2 & -3 & 8 & -1 & 1 \\ -1 & 7 & 2 & -7 & 0 \\ 3 & -8 & -7 & 10 & -1 \\ 2 & -1 & -5 & 3 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 1 & -12 & 8 \\ 0 & 0 & 0 & 103 & -88 \\ 0 & 0 & 0 & 103 & -88 \end{vmatrix} = 0.$$

6. Self-study tasks

Exercise 1. Calculate the determinants of the given second-order matrices.

a) $\mathbf{A} = \begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$, **b)** $\mathbf{B} = \begin{bmatrix} -3 & \sqrt{3} \\ 2\sqrt{2} & \sqrt{6} \end{bmatrix}$, **c)** $\mathbf{C} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

Exercise 2. Calculate the determinants of the given third-order matrices.

a) $\mathbf{A} = \begin{bmatrix} 2 & \frac{1}{3} & -\frac{7}{2} \\ 0 & \sqrt{2} & 15 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$, **b)** $\mathbf{B} = \begin{bmatrix} 3 & 2 & -4 \\ 0 & 5 & 0 \\ -2 & 2 & 2 \end{bmatrix}$, **c)** $\mathbf{C} = \begin{bmatrix} -3 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Exercise 3. Calculate the determinants of the given matrices.

a) $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, **b)** $\mathbf{B} = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 1 & 2 \\ 1 & 2 & 2 & -2 \\ -2 & 2 & 3 & -3 \end{bmatrix}$, **c)** $\mathbf{C} = \begin{bmatrix} 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & -2 \\ -3 & 2 & 1 & -2 \\ 3 & 1 & -1 & 0 \end{bmatrix}$.

Exercise 4. Calculate the determinants of the given matrices.

a) $\mathbf{A} = \begin{bmatrix} 5 & 3 & -1 & 2 & 4 \\ 0 & 2 & 4 & -2 & 2 \\ 0 & 2 & 0 & 3 & -3 \\ 0 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, **b)** $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 2 & 4 & -2 & 4 & 2 \end{bmatrix}$, **c)** $\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 2 & 0 \\ 3 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$.

Exercise 5. Calculate the determinants:

a) $\begin{vmatrix} x+y & x-y \\ x+y & 2x+2y \end{vmatrix}$, **b)** $\begin{vmatrix} a & -1 & 1 \\ a^2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$, **c)** $\begin{vmatrix} 2x & -4x & 1 \\ -2 & 7 & x \\ 1 & -3 & 4 \end{vmatrix}$,

d) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$, **e)** $\begin{vmatrix} 0 & 0 & x & 1 \\ x & 0 & 0 & x \\ 0 & 0 & 0 & x^2 \\ 0 & x & 0 & x^3 \end{vmatrix}$, **f)** $\begin{vmatrix} 1 & 2 & 0 & -1 \\ x & x & 2x & -x \\ 0 & -2 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix}$.

Additional self-solving tasks, with answers and solutions, can be found in the [4].

7. Answers

Answer 1.

a) $\det \mathbf{A} = 31$, **b)** $\det \mathbf{B} = -5\sqrt{6}$, **c)** $\det \mathbf{C} = \cos^2\alpha + \sin^2\alpha = 1$.

Answer 2.

a) $\det \mathbf{A} = \sqrt{2}$, **b)** $\det \mathbf{B} = -10$, **c)** $\det \mathbf{C} = -30$.

Answer 3.

a) $\det \mathbf{A} = 18$, **b)** $\det \mathbf{B} = -57$, **c)** $\det \mathbf{C} = 12$.

Answer 4.

a) $\det \mathbf{A} = 5! = 120$, **b)** $\det \mathbf{B} = 0$, **c)** $\det \mathbf{C} = 6$.

Answer 5.

a) $(x+y)(x+3y)$, **b)** $3a+3a^2$, **c)** $2x^2+24x-1$,

d)

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \overset{ECO2}{=} \ldots = abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{ECO3}{=} \ldots = abc \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \ldots =$$
$$abc(b-a)(c-a)(c-b) = ab^2c^3 + a^2b^3c + a^3bc^2 - ab^3c^2 - a^2bc^3 - a^3b^2c,$$

e) $$\begin{vmatrix} 0 & 0 & x & 1 \\ x & 0 & 0 & x \\ 0 & 0 & 0 & x^2 \\ 0 & x & 0 & x^3 \end{vmatrix} \overset{ECO1}{=} \ldots = -\begin{vmatrix} x & 0 & 0 & 1 \\ 0 & x & 0 & x \\ 0 & 0 & x & x^3 \\ 0 & 0 & 0 & x^2 \end{vmatrix} = -x^5,$$

f) $$\begin{vmatrix} 1 & 2 & 0 & -1 \\ x & x & 2x & -x \\ 0 & -2 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} \overset{ECO1}{=} x \begin{vmatrix} 1 & 2 & 0 & -1 \\ 1 & 1 & 2 & -1 \\ 0 & -2 & 1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} \overset{ECO3}{=} \ldots = -3x.$$

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