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Beata SIKORA

Faculty of Applied Mathematics, Silesian University of Technology

Remarks on the Caputo fractional derivative

Abstract. The purpose of the paper is to familiarise the reader with the concept of the Caputo fractional derivative. The definition and basic properties of the Caputo derivative are given. Formulas for the derivatives of selected functions are derived. Examples of calculating the derivatives of basic functions are presented. The paper also contains a number of self-solving exercises, with answers.
Keywords: fractional derivative, Caputo derivative, fractional calculus.

1. Introduction

Fractional calculus is a rapidly growing branch of mathematics in recent decades. In [14] Bertram Ross gives a historical outline of the theory of differential calculus. Since September 1695, when Leibniz answered de l'Hospital's question "What would we get if in the n-th derivative $\frac{d^n f(x)}{dx^n}$ of the function $f(x) = x$, n were equal to $\frac{1}{2}$?" by writing "It will lead to a paradox. From this apparent paradox, one day useful consequences will be drawn.", many definitions of fractional order derivatives have been formulated. But their correspondence on this subject is considered the beginning of the fractional differential calculus.

In 1819, Sylvestre F. Lacroix gave a formula for the derivative of order $\frac{1}{2}$ for the function $f(x) = x$, namely $\frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$. This derivative was obtained by calculation for $y = x^m$, $m \in \mathbb{N}$:

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}\, x^{m-n},\ m \geq n,$$

and using the gamma function

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\, x^{m-n},$$

in the case of $m = 1$ and $n = \frac{1}{2}$.

In 1823, Niels Abel formulated his theory of fractional order derivatives and integrals and presented its practical application to the solution of the Tautochron problem. This problem involves determining the equation of a curve along which a point mass rolls to the lowest point of the curve in the same time under the influence of a constant gravity, regardless of the starting point on the curve. Joseph Liouville [10,15]

Corresponding author: B. Sikora (Beata.Sikora@polsl.pl).
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made the first major attempt to give a formal definition of a fractional derivative. In 1847, Bernhard Riemann wrote a paper (published posthumously) in which he gives a definition of the fractional order differential operator, probably inspired by Liouville's results [15].

In the second half of the 19th century, it was explicitly stated that a fractional-order derivative must satisfy the following conditions [11, 12, 14]:

(i) Fractional-order derivatives are linear operators.

(ii) The zero-order fractional derivative of the function f is itself a function f.

(iii) If the order of the fractional derivative is a positive integer, then the fractional derivative of this order gives the same result as in the case of the classical derivative.

(iv) The index law for fractional derivatives of arbitrary order holds.

(v) Derivatives of fractional order satisfy the generalized Leibniz rule.

First fractional order derivatives satisfying these conditions are, named in honor of Riemann and Liouville, the Riemann-Liouville fractional derivative and the Grünwald-Letnikov derivative. Formal definitions of the derivatives can be found, among others, in [7, 8, 13, 16].

In 1967, Michele Caputo, in solving certain boundary value problems arising in the theory of viscoelasticity, formulated a new definition of the fractional order derivative [2]. The main advantage of the Caputo approach is that the initial and boundary conditions for differential equations with the Caputo fractional derivative are analogous to the case of integer order differential equations, so they can be interpreted in the same way. Therefore, it is often used in practical applications.

In 2000, Rudolf Hilfer proposed one of a newest definition of fractional derivative [5]. The two-parameter family of the Hilfer fractional derivatives is a generalization of both the Caputo derivative and the Riemann-Liouville derivative. The Hilfer approach allows us to interpolate between these two derivatives of fractional order.

In this paper, the Caputo fractional order derivative is studied. Some properties of the derivative are presented and proved. The Caputo derivatives of selected functions are derived. Numerical examples are also provided.

2. Definition of the Caputo fractional order derivative

Let f be continuous on a closed interval $\langle 0,t\rangle$, $t>0$, i.e. $f\in C(\langle 0,t\rangle)$. From the Cauchy formula for multiple integrals

$$\int_0^t\left(\int_0^{\tau_n}\left(\dots\left(\int_0^{\tau_2} f(\tau_1)d\tau_1\right)\dots\right)d\tau_{n-1}\right)d\tau_n=\frac{1}{(n-1)!}\int_0^t(t-\tau)^{n-1}f(\tau)d\tau,\ \ n\in\mathbb{N} \tag{1}$$

it follows the definition of an integral of an arbitrary order $\alpha\in\mathbb{R}_+$.

Definition 1. *If $f\in C(\langle 0,t\rangle)$, $t\in\mathbb{R}_+$, and $\alpha\in\mathbb{R}_+$, then*

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau \tag{2}$$

is called the Riemann-Liouville fractional integral of order α.

Example 1. Calculate the integral of the function $f(t) = \ln 2$ of order $\alpha = \frac{1}{2}$.
Using formula (2), we get the following

$$I^{\frac{1}{2}} \ln 2 = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-\tau)^{\frac{1}{2}-1} \ln 2 \, d\tau = \frac{\ln 2}{\Gamma(\frac{1}{2})} \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau = \frac{\ln 2}{\Gamma(\frac{1}{2})} \left[-2\sqrt{t-\tau}\right]_0^t = \frac{2\ln 2}{\sqrt{\pi}} \sqrt{t}.$$

Example 2. Calculate the integral of the function $f(t) = e^t$ of order $\alpha = \frac{1}{2}$.

$$I^{\frac{1}{2}} e^t = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{e^\tau}{\sqrt{t-\tau}} d\tau = \left| \begin{array}{l} u = \sqrt{t-\tau} \\ u^2 = t-\tau, \; t-\tau > 0 \\ t - u^2 = \tau \\ -2u du = d\tau \\ \begin{array}{c|c|c} \tau & 0 & t \\ \hline u & \sqrt{t} & 0 \end{array} \end{array} \right| =$$

$$= \frac{-2}{\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{e^{t-u^2}}{u} u du = \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} \, du = \frac{2e^t}{\sqrt{\pi}} \left[\frac{\sqrt{\pi} erf(u)}{2} \right]_0^{\sqrt{t}} = e^t erf(\sqrt{t}),$$

where erf is the error function.

The definition of fractional integral was used in Caputo's definition of fractional derivative.

Definition 2. *If* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *and* $\alpha \in \langle n-1, n)$, $n \in \mathbb{N}$, *then*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau, \tag{3}$$

where Γ *is the gamma function, is called the Caputo fractional derivative of order* α*, provided it exists.*

In the definition of the fractional order derivative according to Caputo, we first calculate the ordinary derivative of natural order and then determine the Riemann-Liouville integral of fractional order from the resulting function. Therefore, for $\alpha \in \langle n-1, n)$, $^{C}D^{\alpha} f(t)$ exists if $f \in C^n(\langle 0, t \rangle)$. Since the Caputo fractional derivative is defined in an integral form, it is a non-local operator. It has "memory" property, which means that a present state depends on past states.

Of course, for $\alpha \to n$ the Caputo derivative tends to n-th order classical derivative of the function f, e.g. $\lim\limits_{\alpha \to n} {}^{C}D^{\alpha} f(t) = f^{(n)}(t)$.

The fractional derivative defined by formula (3) is well-defined, since it satisfies all conditions (i)–(v). The following theorems support this statement.

Let be given functions $f, g\colon \mathbb{R}_+ \to \mathbb{R}$. Assume that the Caputo derivatives of an order $\alpha \in \langle n-1, n)$, $n \in \mathbb{N}$, of the functions exist.

Theorem 1. *The Caputo derivative is a linear operator, i.e. for any* $a, b \in \mathbb{R}$

$$^{C}D^{\alpha}(af(t) + bg(t)) = a\, {}^{C}D^{\alpha} f(t) + b\, {}^{C}D^{\alpha} g(t).$$

Proof. Let ${}^{C}D^{\alpha}f(t)$, ${}^{C}D^{\alpha}g(t)$ be the Caputo derivatives of functions f and g, respectively. Then

$$\begin{aligned}
{}^{C}D^{\alpha}(af(t)+bg(t)) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(af(t)+bg(t))^{(n)}}{(t-\tau)^{\alpha-n+1}}d\tau = \\
&= \frac{1}{\Gamma(n-\alpha)}\left(a\int_0^t \frac{f^{(n)}(t)}{(t-\tau)^{\alpha-n+1}}d\tau + b\int_0^t \frac{g^{(n)}(t)}{(t-\tau)^{\alpha-n+1}}d\tau\right) = \\
&= \frac{1}{\Gamma(n-\alpha)}a\int_0^t \frac{f^{(n)}(t)}{(t-\tau)^{\alpha-n+1}}d\tau + \frac{1}{\Gamma(n-\alpha)}b\int_0^t \frac{g^{(n)}(t)}{(t-\tau)^{\alpha-n+1}}d\tau = \\
&= a\,{}^{C}D^{\alpha}f(t) + b\,{}^{C}D^{\alpha}g(t).
\end{aligned}$$

□

Theorem 2. *The Caputo fractional derivative of order $\alpha = 0$ for f is equal to f, i.e. ${}^{C}D^{0}f(t) = f(t)$.*

Proof. Let the Caputo derivative of order $\alpha = 0$ for f exist. Hence, we have $n = 1$ and from (3) it follows

$$ {}^{C}D^{0}f(t) = \frac{1}{\Gamma(1)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{-1+1}}d\tau = \int_0^t f'(\tau)d\tau = f(t). $$

□

Theorem 3. *If $\alpha = k$ for some $k \in \mathbb{N}$, then ${}^{C}D^{\alpha}f(t) = f^{(k)}(t)$ for $t \in \mathbb{R}_+$.*

Proof. Suppose there exists the Caputo derivative of an order $\alpha \in \mathbb{N}$ for f. Since, from Def. 2, $\alpha \in \langle n-1, n)$, it follows that $\alpha = n-1$, i.e. $k = n-1$. Hence

$$\begin{aligned}
{}^{C}D^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau = \frac{1}{\Gamma(n-n+1)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-1-n+1}}d\tau = \\
&= \frac{1}{\Gamma(1)}\int_0^t f^{(n)}(\tau)d\tau = f^{(n-1)}(t) = f^{(k)}(t).
\end{aligned}$$

□

Theorem 4. *The index law for the Caputo derivative of arbitrary order holds, i.e.*

$$ {}^{C}D^{\alpha}\;{}^{C}D^{\beta}f(t) = {}^{C}D^{\alpha+\beta}f(t) $$

for any $\alpha, \beta \in \langle n-1, n)$, $n \in \mathbb{N}$.

Theorem 4 is also called the semigroup property of the Caputo fractional operator ${}^{C}D^{\alpha}$.

Theorem 5. *The Caputo fractional derivative satisfies generalized Leibniz rule, i.e. for* $\alpha \in \langle n-1, n)$

$$ {}^{C}D^{\alpha}\big(f(t)g(t)\big) = \sum_{i=1}^{\infty} \binom{\alpha}{i} \, {}^{C}D^{i} f(t) \, {}^{C}D^{\alpha-i} g(t). $$

Theorems 4 and 5 are presented here without proofs. The proofs are based on the relation between Caputo and Riemann-Liouville fractional differential operators, which is not mentioned in this study. Readers interested in these proofs are referred to [4].

3. The Caputo fractional derivatives of selected functions

In this section the Caputo fractional derivatives of selected functions are derived and illustrative examples are presented. Let $t \in \mathbb{R}_+$ and $\alpha \in \langle n-1, n)$, $n \in \mathbb{N}$.

Constant function

$$ f(t) = c, \; c \in \mathbb{R} $$

For the function $f(t) = c$, $c \in \mathbb{R}$, directly from Def. 2, we have

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{0}{(t-\tau)^{\alpha-n+1}} \, d\tau = 0. $$

Precisely because the Caputo derivative of a real constant is always equal to 0, the initial conditions in Cauchy problems with the Caputo derivative can be interpreted analogously to the classical integer order derivative.

Power function

$$ f(t) = t^p, \; p \in \mathbb{R} $$

Before deriving a general form of the Caputo derivative for power functions, we propose two examples.

Example 3. By definition, calculate the derivative ${}^{C}D^{\frac{1}{2}} t$.

$$ {}^{C}D^{\frac{1}{2}} t = \frac{1}{\Gamma(1-\frac{1}{2})} \int\limits_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau = \left| \begin{array}{l} u = t-\tau \\ du = -d\tau \\ d\tau = -du \\ \begin{array}{c|c|c} \tau & 0 & t \\ \hline u & t & 0 \end{array} \end{array} \right| = $$

$$ = \frac{1}{\sqrt{\pi}} \int\limits_t^0 \frac{-1}{\sqrt{u}} du = \frac{1}{\sqrt{\pi}} \int\limits_0^t \frac{1}{\sqrt{u}} du = \frac{1}{\sqrt{\pi}} \left[2\sqrt{u} \right]_0^t = \frac{1}{\sqrt{\pi}} (2\sqrt{t} - 2\sqrt{0}) = \frac{2\sqrt{t}}{\sqrt{\pi}}. $$

Example 4. By definition, calculate the derivative ${}^CD^{\frac{1}{2}}(t^2+2)$.

$$
{}^CD^{\frac{1}{2}}(t^2+2) = \frac{1}{\Gamma(\frac{1}{2})}\int\limits_0^t \frac{2\tau}{\sqrt{t-\tau}}d\tau = \left| \begin{array}{l} u=\sqrt{t-\tau} \\ u^2 = t-\tau,\ t-\tau>0 \\ t-u^2=\tau \\ -2udu = d\tau \\ \begin{array}{c|c|c} \tau & 0 & t \\ \hline u & \sqrt{t} & 0 \end{array} \end{array} \right| = \frac{1}{\sqrt{\pi}}\int\limits_{\sqrt{t}}^0 \frac{-4(t-u^2)}{u}udu
$$

$$
= \frac{4}{\sqrt{\pi}}\int\limits_0^{\sqrt{t}}(t-u^2)du = \frac{4}{\sqrt{\pi}}\left[ut - \frac{u^3}{3}\right]_0^{\sqrt{t}} = \frac{4}{\sqrt{\pi}}\left(t\sqrt{t} - \frac{(\sqrt{t})^3}{3}\right) = \frac{8t\sqrt{t}}{3\sqrt{\pi}} = \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}.
$$

From Def. 3 the general form of the derivative of a power function can be derived.

The Caputo fractional derivative of the power function is defined by the following formula

$$
{}^CD^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}, & p>n-1,\ p\in\mathbb{R} \\ 0, & p\le n-1,\ p\in\mathbb{N}. \end{cases} \tag{4}
$$

To derive formula (4) (see also [6]), we start with $p>n-1$, $p\in\mathbb{R}$.

$$
{}^CD^\alpha t^p = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(\tau^p)^{(n)}}{(t-\tau)^{\alpha-n+1}}\,d\tau = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\Gamma(p+1)}{\Gamma(p-n+1)}\tau^{p-n}(t-\tau)^{n-\alpha-1}\,d\tau.
$$

Using the substitution $\tau=\lambda t$ for $0\le\lambda\le 1$, we can rewrite the derivative in the form

$$
{}^CD^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}\int_0^t (\lambda t)^{p-n}((1-\lambda)t)^{n-\alpha-1}\,d\lambda
$$

$$
= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}t^{p-\alpha}\int_0^t \lambda^{p-n}(1-\lambda)^{n-\alpha-1}\,d\lambda.
$$

After applying the beta function and its representation by the gamma function, we get

$$
{}^CD^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}t^{p-\alpha}B(p-n+1,n-\alpha)
$$

$$
= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}t^{p-\alpha}\frac{\Gamma(p-n+1)\Gamma(n-\alpha)}{\Gamma(p-\alpha+1)} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}.
$$

For $p\le n-1, p\in\mathbb{N}$, $(t^p)^{(n)}=0$, and hence ${}^CD^\alpha t^p = 0$.

Example 5. Calculate the Caputo derivative of order $\alpha=\frac{2}{3}$ for the function $f(t)=t^3$.

We have $p=3$ and $n=1$. Therefore, by (4), we have

$$
{}^CD^{\frac{2}{3}}t^3 = \frac{\Gamma(4)}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}} = \frac{3!}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}} = \frac{6}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}}.
$$

Exponential function

$$f(t) = e^{at},\ a \in \mathbb{R}$$

The Caputo fractional derivative of the exponential function is defined by the following formula

$$^{C}D^{\alpha}e^{at} = \sum_{k=0}^{+\infty} \frac{a^{k+n}t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = a^n t^{n-\alpha} E_{1,n-\alpha+1}(at), \tag{5}$$

where $E_{\alpha,\beta}$ is the two-parameter Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta \in \mathbb{R}_+, z \in \mathbb{C}.$$

The proof of the formula (5) requires the use of the Riemann-Liouville derivative, so it is omitted here. It can be found in [6].

Example 6. Calculate the Caputo derivative of order $\alpha = \frac{1}{2}$ for the function $f(t) = e^t$.
By (5), we have

$$^{C}D^{\frac{1}{2}}e^{t} = t^{\frac{1}{2}} E_{1,\frac{3}{2}}(t) = \sqrt{t}\, E_{1,\frac{3}{2}}(t).$$

Let us calculate the Mittag-Leffler function.

$$E_{1,\frac{3}{2}}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(k+\frac{3}{2})} = \frac{1}{t}\sum_{k=0}^{+\infty} \frac{t^{k+1}}{\Gamma(k+1+\frac{1}{2})} = \frac{1}{t}\sum_{k=0}^{+\infty} \frac{t^{k+1}}{\frac{\sqrt{\pi}(2k+1)!!}{2^{k+1}}} = \frac{1}{t\sqrt{\pi}}\sum_{k=0}^{+\infty} \frac{t^{k+1}2^{k+1}}{(2k+1)!!}$$

$$= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{+\infty} \frac{t^k 2^k}{(2k+1)!!} = \frac{e^t\ \mathrm{erf}(\sqrt{t})}{\sqrt{t}},$$

where $\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} \sum_{n=0}^{+\infty} \frac{2^n}{(2n+1)!!} t^{2n+1}$ is the error function. It follows that

$$^{C}D^{\frac{1}{2}}e^{t} = t^{\frac{1}{2}} E_{1,\frac{3}{2}}(t) = \sqrt{t}\,\frac{e^t\ \mathrm{erf}(\sqrt{t})}{\sqrt{t}} = e^t\ \mathrm{erf}(\sqrt{t}).$$

4. Exercises

For the reader who wishes to try his hand at calculating integrals and fractional derivatives, several self-solving exercises are suggested in this chapter. The answers to the exercises are also given.

Exercise 1. Calculate the integral of the function $f(t) = 2$ of order $\alpha = \frac{1}{2}$.

Answer: $I^{\frac{1}{2}}2 = \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}}$.

Exercise 2. Calculate the integral of the function $f(t) = t^2$ of order $\alpha = \frac{1}{2}$.

Answer: $I^{\frac{1}{2}}t^2 = \frac{16}{15\sqrt{\pi}} t^{\frac{5}{2}}$.

Exercise 3. Calculate the Caputo derivative of order $\alpha = \frac{3}{4}$ for the function $f(t) = t$.

Answer: ${}^{C}D^{\frac{3}{4}} t = \frac{1}{\Gamma(\frac{5}{4})} t^{\frac{1}{4}}$.

Exercise 4. Calculate the Caputo derivative of order $\alpha = \frac{1}{3}$ for the function $f(t) = t^2$.

Answer: ${}^{C}D^{\frac{1}{3}} t^2 = \frac{2}{\Gamma(\frac{8}{3})} t^{\frac{5}{3}}$.

Exercise 5. Calculate the Caputo derivative of order $\alpha = \frac{1}{2}$ for the function $f(t) = e^{2t}$.

Answer: ${}^{C}D^{\frac{1}{2}} e^{2t} = \sqrt{2}\, e^{2t} \operatorname{erf}(\sqrt{2t})$.

5. Final remarks

Fractional order differentiation is the generalization of integer order differentiation. As the reader can see, the computation of fractional order derivatives is much more complicated than that of classical integer order derivatives. However, it has been shown that mathematical models based on fractional order integrals and derivatives describe the properties of many phenomena more accurately than the previously used integer order models. This is due to the fact that systems are usually not perfect and can be, for example, perturbed by external forces. Therefore, integer order derivatives may not be appropriate for understanding the trajectories of state variables. With fractional derivatives, we have an infinite number of derivative orders at our disposal, so we can determine which fractional differential equation better describes the dynamics of the model. Experimental data and algorithms for certain real-world phenomena showing that fractional order derivatives provide more efficient modelling of the solution curve are presented in [1] and [3], among others.

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